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Dynamical Casimir Effect for Scalar Fields II (Energy Calculation)

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The energy of the evolved vacuum state is calculated. From a frequency cut-off regularisation the divergent terms are separated and, in the $1 + 1$ dimensional case they are removed with a mass renormalisation of the moving boundary. A renormalisation of the external force is also needed in $3 + 1$ dimensions.

KEY WORDS: dynamical casimir effect; frequency cut-off regularisation; back-reaction.

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1. INTRODUCTION

In the preceding paper hereafter referred as I, we have considered the creation of particles in domains with moving boundaries. In the present paper we study the energy of the evolved vacuum state. We show that this energy is splinted in two parts, the energy of the vacuum state that after renormalisation becomes the standard Casimir energy, and a dynamical part. For times greater than the stopping time, i.e. when the boundary returns to the initial position, the dynamical part is the energy of the produced particles, that is, the total radiated energy, which is finite if the movement of the boundary is smooth enough. On the other hand, when the boundary moves the dynamical energy has some divergent terms.

To obtain the renormalised dynamical energy, we have used the frequency cut-off regularisation. The desired result is obtained after a mass renormalisation in the $1 + 1$-dimensional case. In the $3 + 1$-dimensional case we have also needed the renormalisation of the external force that produces the movement of the boundary. Once we have calculated the dynamical energy, we show that, when the boundary moves, this quantity is not positive. This proves that, when the boundary is not at rest, the dynamical energy cannot be considered as the radiated energy in

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agreement with the fact that, in this situation, the concept of particle is ill-defined (see the discussion below to the formula (25) of I).

Finally we consider a semi-classical model of the interaction between a moving boundary and the Klein-Gordon field, i.e., a model that describes the particle creation process due to the movement of the boundary, and that also takes into account the movement of the boundary induced by the radiation (the back-reaction).

The paper is organised as follows:

In Section 2 we show that the energy of the evolve vacuum can be splinted in a static part (the Casimir energy) and a dynamical one (the energy of the created "quasi-particles").

In Section 3 we consider the $1 + 1$-dimensional case. Firstly, using the frequency cut-off regularisation, we study the energy when there is a single moving mirror. Once we have calculated the renormalised physical quantities we compare our results with those obtained by Davis and Fulling in Fulling and Davies (1976). We also consider bounded cavities, in this case we obtain the main result of the paper, the renormalised dynamical energy when it exist two mirrors.

In Section 4 we study the $3 + 1$-dimensional case. For a single moving mirror we calculate in a very easy way the renormalised energy of the vacuum state and the renormalised dynamical energy of the evolved vacuum state that coincides with the result obtained by Ford and Vilenkin (1982).

Finally in Section 5 we study a semi-classical model of the interaction between one single moving mirror and the Klein-Gordon field, that includes the recoil of the mirror.

2. THE CASIMIR ENERGY

Let $\langle \hat{E}^m(t) \rangle$ be the energy of the evolved vacuum state inside a cavity Ω_t. From the formulae (18) and (19) of I, we have

$$\langle \hat{E}^m(t) \rangle \equiv \langle 0|(\mathcal{T}^t)^\dagger \hat{E}(t)\mathcal{T}^t|0\rangle = \sum_{\mathbf{n}} \frac{\hbar}{2}\omega_{\mathbf{n}}(t;\epsilon) + \sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}}(t;\epsilon)\mathcal{N}_{\mathbf{n}}^m(t) + \mathcal{O}(\epsilon^4). \qquad (1)$$

This formula shows that the energy is splinted in two parts, the energy of the vacuum at time t that, after renormalisation, gives the usual Casimir energy, and a dynamical part that corresponds to the energy of the created "quasi-particles" in the cavity.

Note that, when $t \geq T$, we have

$$\langle \hat{E}^m(t \geq T)\rangle = \sum_{\mathbf{n}} \frac{\hbar}{2}\omega_{\mathbf{n}}(0) + \sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}}(0)\mathcal{N}_{\mathbf{n}}^m(t \geq T) + \mathcal{O}(\epsilon^4). \quad (2)$$

That is, when the boundary returns to the initial position, the energy is decomposed in two parts, the energy of the initial vacuum state and the radiated energy (the energy of the created particles into the cavity).

Example 2.1. For a rectangular cavity $\Omega_t = [0, L_1 + \epsilon g(t)] \times [0, L_2] \times [0, L_3]$ the formula (1) behaves

$$\begin{aligned}\langle \hat{E}^m(t)\rangle = &\sum_{\mathbf{n}\in\mathbb{N}^3} \frac{\hbar}{2}\omega_{\mathbf{n}}(t;\epsilon) \\ &+ \frac{\epsilon^2\hbar}{2{L_1}^2}\left(\frac{c\pi}{L_1}\right)^4 \\ &\times \sum_{\mathbf{k},\mathbf{n}\in\mathbb{N}^3} \frac{k_1^2 n_1^2 \left|\int_0^t \dot{g}(\tau)e^{i(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))\tau}d\tau\right|^2}{\omega_{\mathbf{n}}(0)\omega_{\mathbf{k}}(0)(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))}\delta_{k_2,n_2}\delta_{k_3,n_3} + \mathcal{O}(\epsilon^4), \quad (3)\end{aligned}$$

3. 1 + 1-DIMENSIONAL CASE

3.1. A Single Moving Mirror

Here we consider a moving mirror following a prescribed trajectory $(\epsilon g(t), t)$. We denote by $\langle \hat{E}^0_{+,\mathrm{dyn}}(t)\rangle$ the dynamical energy on the right side of the mirror, and by $\langle \hat{E}^0_{-,\mathrm{dyn}}(t)\rangle$ the dynamical energy on the left one.

Until order ϵ^2, the dynamical energy at both sides of the mirror, can be calculated using the eigenfunctions $f_{\pm,n}(t,x;\epsilon) = \sqrt{\frac{2}{L\mp\epsilon g(t)}}\sin(n\pi\frac{x-\epsilon g(t)}{L\mp\epsilon g(t)})$ and taken $L \to \infty$ or directly using the continuous set of eigenfunctions $f_\omega(t,x;\epsilon) = \sqrt{\frac{2}{\pi c}}\sin(\frac{\omega}{c}(x-\epsilon g(t))$. The result is

$$\left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t)\right\rangle = \frac{\epsilon^2\hbar}{12c^2\pi^2}\int_0^\infty d\omega\omega^2\left|\int_0^t \dot{g}(\tau)e^{i\omega\tau}d\tau\right|^2. \quad (4)$$

The energy of the vacuum at time t, on the left and on the right side, is

$$\left\langle \hat{E}^0_{\pm,vac}(t)\right\rangle \equiv \lim_{L\to\infty}\sum_{n=1}^\infty \frac{\hbar}{2}\omega_{\pm,n}(t;\epsilon), \quad (5)$$

where we have introduced the frequencies $\omega_{\pm,n}(t;\epsilon) = \frac{n\pi c}{L\mp\epsilon g(t)}$.

Using the frequency cut-off $e^{-\omega\gamma}$ with $0 < \gamma \ll 1$, we define the regularised dynamical energy by

$$\left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t;\gamma)\right\rangle \equiv \lim_{L\to\infty}\sum_{n=1}^{\infty}\hbar\omega_{\pm,n}(t;\epsilon)\mathcal{N}^0_{\pm,n}(t)e^{-\gamma\omega_{\pm,n}(t;\epsilon)}. \tag{6}$$

If we assume that $g \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^3[0,T]$, we obtain the same result as Barton and Eberlein (1994) (see formula (7.17b) of the Barton and Eberlein (1994) in the perfect-reflector limit), i.e., under order ϵ^2 then regularised dynamical energy is

$$\left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t;\gamma)\right\rangle = \frac{\epsilon^2\hbar}{12c^2\pi}\left[\frac{\dot{g}^2(t)}{\pi\gamma} - \ddot{g}(t)\dot{g}(t) + \int_0^t \dddot{g}^2(\tau)\,d\tau\right] \tag{7}$$

Thus, to renormalise the total dynamical energy we must impose that the kinetic energy of the moving boundary is

$$\frac{1}{2}\left(M_{\mathrm{exp}} - \frac{\hbar}{3c^2\pi^2\gamma}\right)\epsilon^2\dot{g}^2(t), \tag{8}$$

where M_{exp} is the experimental mass of the moving boundary. Then, the renormalised dynamical energy, namely $\langle\hat{\mathbb{E}}^0_{\pm,\mathrm{dyn}}(t)\rangle$, is

$$\left\langle \hat{\mathbb{E}}^0_{\pm,\mathrm{dyn}}(t)\right\rangle \equiv \frac{\epsilon^2\hbar}{12c^2\pi}\left[-\ddot{g}(t)\dot{g}(t) + \int_0^t \dddot{g}^2(\tau)\,d\tau\right]. \tag{9}$$

Remark 3.1. When $t \geq T$ we have

$$\left\langle \hat{\mathbb{E}}^0_{\pm,\mathrm{dyn}}(t\geq T)\right\rangle = \left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t\geq T)\right\rangle = \frac{\epsilon^2\hbar}{12c^2\pi}\int_0^T \dddot{g}^2(\tau)\,d\tau > 0, \tag{10}$$

in agreement with Ford and Vilenkin (1982) and Schützhold *et al.* (1998).

On the other hand, when $t \leq \delta$, with $0 < \delta \ll 1$ we have $\langle\hat{\mathbb{E}}^0_{\pm,\mathrm{dyn}}(t)\rangle < 0$. This shows that, when the boundary moves, the dynamical energy cannot be considered as the radiated energy until time t. (see for details the paragraph below to the formula (4.5) of Fulling and Davies (1976))

Note that we can write

$$\mathcal{N}^0_{\pm}(t\geq T) = \int_0^{\infty}\mathcal{N}^0_{\pm,\omega}(t\geq T)\,d\omega$$

$$\left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t\geq T)\right\rangle = \int_0^{\infty}\hbar\omega\mathcal{N}^0_{\pm,\omega}(t\geq T)\,d\omega,$$

where, until order ϵ^2,

$$\mathcal{N}^0_{\pm,\omega}(t \geq T) \equiv \frac{\epsilon^2}{\pi^2 c^2} \int_0^{\infty} \frac{\omega\omega'}{(\omega+\omega')^2} \left| \int_0^T \dot{g}(\tau) e^{i(\omega+\omega')\tau} d\tau \right|^2 d\omega', \tag{11}$$

is the average density of produced particles per unit of frequency on the right and on the left side of the mirror. Consequently, $\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t \geq T)\rangle$ is the radiated energy at both sides, i.e., the energy of the produced particles.

To calculate the energy of the vacuum, we use the same frequency cut-off. It is not difficult to find

$$\left\langle \hat{E}^0_{\pm,\mathrm{vac}}(t;\gamma)\right\rangle \equiv \lim_{L\to\infty} \sum_{n=1}^{\infty} \frac{\hbar}{2} \omega_{\pm,n}(t;\epsilon) e^{-\omega_{\pm,n}(t;\epsilon)\gamma} = \lim_{L\to\infty} \frac{\hbar}{2c\pi\gamma^2}(L \mp \epsilon g(t)),$$

and thus, the energy of the vacuum per unit length, whose value is $\frac{\hbar}{2c\pi\gamma^2}$, coincides with the energy density of the Minkowskian vacuum. Consequently, the energy of the vacuum on the right side (resp. on the left side) can be removed subtracting the energy of the Minkowskian vacuum inside the domain $[\epsilon g(t), \infty)$ (resp. $(-\infty, \epsilon g(t)]$).

We have also calculated, until order ϵ, the energy density in both sides of the mirror. We have found

$$\left\langle \hat{\mathcal{E}}^0_{\pm}(t,x;\gamma)\right\rangle = \frac{\hbar}{2c\pi\gamma^2} \mp \frac{\epsilon\hbar}{12c^2\pi}\left(\dddot{g}\left(t \mp \frac{x}{c}\right) - \frac{\ddot{g}(t)}{\pi} \frac{2\gamma}{\gamma^2 + \frac{x^2}{c^2}} \right). \tag{12}$$

In a distributional sense, we can write

$$\left\langle \hat{\mathcal{E}}^0_{\pm}(t,x;\gamma)\right\rangle = \frac{\hbar}{2c\pi\gamma^2} \mp \frac{\epsilon\hbar}{12c^2\pi}\left(\dddot{g}\left(t \mp \frac{x}{c}\right) - 2c\,\ddot{g}(t)\delta(x) \right). \tag{13}$$

Remark 3.2. Note that, until order ϵ, we have

$$\int_{\epsilon g(t)}^{\pm\infty} \frac{\epsilon\hbar}{12c^2\pi}\left(\dddot{g}\left(t \mp \frac{x}{c}\right) - \frac{\ddot{g}(t)}{\pi} \frac{2\gamma}{\gamma^2 + \frac{x^2}{c^2}} \right) dx = 0,$$

in agreement with the fact that energy does not contain terms of order ϵ.

Note also that, from the Appendix B, we can see that the radiation reaction force is

$$\frac{\epsilon\hbar}{6c^2\pi}\left(\dddot{g}(t) - \frac{2\ddot{g}(t)}{\pi\gamma} \right).$$

Finally we compare our results with those obtained by Davis and Fulling in Fulling and Davies (1976). D-F consider the problem

$$\begin{cases} \phi_{tt} - c^2\phi_{xx} = 0 & \forall x > \epsilon g(t)\ \forall t \in \mathbb{R} \\ \phi(t, \epsilon g(t)) = 0 & \forall t \in \mathbb{R}. \end{cases} \tag{14}$$

The complete set of in-going positive-frequency solutions of (14) is given by (see DeWitt, 1975)

$$\phi_\omega(t, x) = \frac{i}{\sqrt{4\hbar\omega\pi}} \left(e^{-i\omega v} - e^{-i\omega p(u)}\right), \tag{15}$$

with $v = t + \frac{x}{c}$, $u = t - \frac{x}{c}$ and $p(u) = 2t(u) - u$ where $t(u)$ is defined by $t(u) = u + \frac{1}{c}\epsilon g(t(u))$.

Remark 3.3. The function $p(u)$ can be expanded in powers of ϵ in the following way: From the system

$$\begin{cases} p(u) = 2t(u) - u \\ t(u) = u + \frac{1}{c}\epsilon g(t(u)), \end{cases} \tag{16}$$

inserting the second equation into the first one, we obtain

$$p(u) = u + \frac{2}{c}\epsilon g(t(u)) = u + \frac{2}{c}\epsilon g(u) + \mathcal{O}(\epsilon^2). \tag{17}$$

Repeating this process once again we get

$$p(u) = u + \frac{2}{c}\epsilon g(u) + \frac{2}{c^2}\epsilon^2 g(u)\dot{g}(u) + \mathcal{O}(\epsilon^3). \tag{18}$$

In the Heisenberg picture, the quantum field on the right side, has the form

$$\hat{\phi}(t, x) = \int_0^\infty d\omega\left[\hat{a}_\omega^{in}\phi_\omega(t, x) + \left(\hat{a}_\omega^{in}\right)^\dagger \phi_\omega^*(t, x)\right], \tag{19}$$

thus, using the "point-splitting" ansatz, D-F show that the regularised energy density on the right side of the mirror is given by

$$\langle\hat{\mathcal{E}}_+^0(t, x; \gamma)\rangle_{DF} = \frac{\hbar}{2c\pi\gamma^2} + \frac{\hbar}{12c\pi}(p'(u))^{1/2}[(p'(u))^{-1/2}].'' \tag{20}$$

Now, to obtain until order ϵ^2 the energy density on the right side, we insert (18) into (20), and we get

$$\langle\hat{\mathcal{E}}_+^0(t, x; \gamma)\rangle_{DF} = \frac{\hbar}{2c\pi\gamma^2} - \epsilon\frac{\hbar}{12c^2\pi}\left(\dddot{g}(u) + \frac{\epsilon}{c}\ddddot{g}(u)g(u) + 2\frac{\epsilon}{c}\dddot{g}(u)\dot{g}(u)\right), \tag{21}$$

where the first term is the energy density of the Minkowskian vacuum.

The dynamical energy on the right side is obtained integrating $\langle \hat{\mathcal{E}}^0_+(t,x;\gamma)\rangle_{DF} - \frac{\hbar}{2c\pi\gamma^2}$ between $\epsilon g(t)$ and ∞. A simple calculation lead to the following result

$$\langle \hat{E}^0_{+,\mathrm{dyn}}(t;\gamma)\rangle_{DF} = \frac{\epsilon\hbar}{12c\pi}\left[-\ddot{g}(t) - \frac{\epsilon}{c}\ddot{g}(t)\dot{g}(t) + \frac{\epsilon}{c}\int_0^t \ddot{g}^2(\tau)\,d\tau\right]. \quad (22)$$

Remark 3.4. The energy and the energy density on the left, can be easily calculated making the change $\epsilon \to -\epsilon$ and $u \to v$.

Some comments arise from these results:

(i) The dynamical energy and the dynamical energy density obtained by D-F taking the set of mode functions (15) and using the regularisation procedure based in the "point-splitting" ansatz, do not have any divergent term. We will see that in $3+1$-dimensional case, some divergent terms appear in the dynamical energy when a set of mode functions is used. Only in the $1+1$ case the dynamical energy is free of divergent terms.

(ii) The difference between (7) and (22) (resp. (12) and (21)), is due to the fact that we have used two different regularisation methods.

(iii) When the boundary returns at rest, (7) and (22) give, of course, the same result (the radiated energy). And also, if we consider the total renormalised energy, both methods give the same result.

(iv) The frequency cut-off regularisation can be used in the massive case, (here there is already a renormalisation prescription (Bordag *et al.*, 2000)). In the Appendix A we prove that the renormalised dynamical energy in the massive case (unambiguously calculated) converges to the result obtained in formula (7). This fact shows the consistency of the frequency cut-off regularisation.

(v) The "point-splitting" ansatz is useful, in practice, when we have a set of mode functions that satisfies exactly the field equation and only approximately the boundary conditions. On the other hand, the frequency cut-off regularisation is a suitable method when we have a perturbative solution of the Schrödinger equation given by the Hamiltonian (16) of I. This shows the great advantage of the frequency cut-off prescription because it is very easy in the massless and in the massive case, to calculate in practice approximate solutions of the Schrödinger equation. Unfortunately, exact solutions of the field equation can only be obtained in particular situations.

3.2. Bounded Cavities

In this Section we consider the following $1+1$-dimensional cavity $\Omega_t = [\epsilon g(t), L]$. For a massless field, the frequency cut-off regularised dynamical energy inside the cavity is:

$$\left\langle \hat{E}^0_{\text{in, dyn}}(t;\gamma)\right\rangle = \frac{\epsilon^2\hbar\pi c}{12L^3}\sum_{n=1}^{\infty}(n^2-1)\left|\int_0^t \dot{g}(\tau)e^{i\frac{c\pi}{L}n\tau}d\tau\right|^2 e^{-\gamma\frac{c\pi}{L}n}. \qquad (23)$$

To obtain a explicit expression of this energy, we will use the theory of Fourier's series (see for a detailed explanation Gasquet and Witomski (1995)).

Let N^* be the natural number that satisfies $\frac{2L}{c}N^* < t \leq \frac{2L}{c}(N^*+1)$. For a given function f, using the Heaviside's step function, namely θ, we can write

$$\int_0^t f(\tau)e^{i\frac{c\pi}{L}n\tau}d\tau = \sqrt{\frac{2L}{c}}\left(\sum_{k=0}^{N^*-1} c_{-n}\left[f_{\frac{2L}{c}k}\right] + c_{-n}\left[(f\theta_t)_{\frac{2L}{c}N^*}\right]\right),$$

where $c_n[f_{\frac{2L}{c}k}]$ is the n-Fourier coefficient of the $\frac{2L}{c}$-periodic function $f_{\frac{2L}{c}k}$, defined in the interval $[0, \frac{2L}{c}]$ by $f_{\frac{2L}{c}k}(\tau) \equiv f(\tau + \frac{2L}{c}k)$, and $c_n[(f\theta_t)_{\frac{2L}{c}N^*}]$ is the n-Fourier coefficient of the $\frac{2L}{c}$-periodic function $(f\theta_t)_{\frac{2L}{c}N^*}$, defined in the interval $[0, \frac{2L}{c}]$ by $(f\theta_t)_{\frac{2L}{c}N^*}(\tau) \equiv f(\tau + \frac{2L}{c}N^*)\theta(t - \tau - \frac{2L}{c}N^*)$.

Then using the set of functions $e_n(t) \equiv \sqrt{\frac{c}{2L}}e^{i\frac{\pi nc}{L}t}$ and integrating by parts, we can find

$$\begin{aligned}\left\langle \hat{E}^0_{\text{in, dyn}}(t;\gamma)\right\rangle &= \frac{\epsilon^2\hbar}{12\pi^2c^2}\frac{\dot{g}^2(t)}{\gamma^2} + \frac{\epsilon^2\hbar\pi c}{24L^3}g^2(t)\\ &\quad + \frac{\epsilon^2\hbar}{12\pi c^2}\sum_{n\in\mathbb{Z}}\left|\sum_{k=0}^{N^*-1} c_n\left[\ddot{g}_{\frac{2L}{c}k}\right] + c_n\left[(\ddot{g}\theta_t)_{\frac{2L}{c}N^*}\right]\right|^2\\ &\quad - \frac{\epsilon^2\hbar\pi}{12L^2}\sum_{n\in\mathbb{Z}}\left|\sum_{k=0}^{N^*-1} c_n\left[\dot{g}_{\frac{2L}{c}k}\right] + c_n\left[(\dot{g}\theta_t)_{\frac{2L}{c}N^*}\right]\right|^2\\ &\quad - \frac{\epsilon^2\hbar}{6\pi c^2}\dot{g}(t)\sum_{n\in\mathbb{Z}}e_n(t)\left(\sum_{k=0}^{N^*-1} c_n\left[\ddot{g}_{\frac{2L}{c}k}\right] + c_n\left[(\ddot{g}\theta_t)_{\frac{2L}{c}N^*}\right]\right). \qquad (24)\end{aligned}$$

Applying the Parseval identity and the Dirichlet theorem (Gasquet and Witomski, 1995) we can deduce that

$$\begin{aligned}\left\langle \hat{E}^0_{\text{in, dyn}}(t;\gamma)\right\rangle &= \frac{\epsilon^2\hbar}{12\pi^2c^2}\frac{\dot{g}^2(t)}{\gamma^2} + \frac{\epsilon^2\hbar\pi c}{24L^3}g^2(t)\\ &\quad + \frac{\epsilon^2\hbar}{6\pi c^2}\left(\frac{1}{2}\int_0^t \ddot{g}^{\,2}(\tau)\,d\tau + \sum_{k=1}^{N^*}\int_0^t \ddot{g}\left(\tau - \frac{2L}{c}k\right)\ddot{g}\,(\tau)\,d\tau\right)\end{aligned}$$

$$- \frac{\epsilon^2 \hbar \pi}{6L^2} \left(\frac{1}{2} \int_0^t \dot{g}^2(\tau) d\tau + \sum_{k=1}^{N^*} \int_0^t \dot{g}\left(\tau - \frac{2L}{c}k\right) \dot{g}(\tau) d\tau \right)$$
$$- \frac{\epsilon^2 \hbar}{6\pi c^2} \dot{g}(t) \left(\sum_{k=1}^{N^*} \ddot{g}\left(t - \frac{2L}{c}k\right) + \frac{1}{2} \ddot{g}(t) \right). \tag{25}$$

Outside Ω_t we obtain the same result as formula (7). Then, the total renormalised dynamical energy, denoted by $\langle \hat{\mathbb{E}}_{\text{tot dyn}}(t) \rangle$, is

$$\langle \hat{\mathbb{E}}_{\text{tot,dyn}}(t) \rangle = \frac{\epsilon^2 \hbar \pi c}{24L^3} g^2(t) + \frac{\epsilon^2 \hbar}{6\pi c^2} \sum_{k=0}^{N^*} \int_0^t \ddot{g}\left(\tau - \frac{2Lk}{c}\right) \ddot{g}(\tau)\, d\tau$$
$$- \frac{\epsilon^2 \hbar \pi}{6L^2} \left(\frac{1}{2} \int_0^t \dot{g}^2(\tau)\, d\tau + \sum_{k=1}^{N^*} \int_0^t \dot{g}\left(\tau - \frac{2Lk}{c}\right) \dot{g}(\tau) d\tau \right)$$
$$- \frac{\epsilon^2 \hbar}{6\pi c^2} \dot{g}(t) \sum_{k=0}^{N^*} \ddot{g}\left(t - \frac{2L}{c}k\right). \tag{26}$$

Integrating by parts, (26) becomes

$$\langle \hat{\mathbb{E}}_{\text{tot,dyn}}(t) \rangle = -\epsilon \int_0^t F_1(\tau) \dot{g}(\tau)\, d\tau, \tag{27}$$

with

$$F_1(t) \equiv -\frac{\epsilon \hbar \pi c}{12L^3} g(t) + \frac{\epsilon \hbar}{6\pi c^2} \sum_{k=0}^{N^*} \ddot{g}\left(t - \frac{2Lk}{c}\right) + \frac{\epsilon \hbar \pi}{6L^2} \left(\frac{1}{2} \dot{g}(t) + \sum_{k=1}^{N^*} \dot{g}\left(t - \frac{2Lk}{c}\right)\right). \tag{28}$$

The frequency cut-off regularisation shows that the renormalised total energy of the vacuum state at time t is

$$\langle \hat{\mathbb{E}}_{\text{tot,vac}}(t) \rangle = -\frac{1}{24} \frac{\hbar \pi c}{L - \epsilon g(t)} = -\frac{1}{24} \frac{\hbar \pi c}{L} - \epsilon \int_0^t F_2(\tau) \dot{g}(\tau)\, d\tau, \tag{29}$$

where

$$F_2(t) \equiv \frac{1}{24} \frac{\hbar \pi c}{(L - \epsilon g(t))^2} = \frac{1}{24} \frac{\hbar \pi c}{L^2} + \frac{1}{12} \frac{\hbar \pi c}{L^3} \epsilon g(t) + \mathcal{O}\left(\frac{\hbar c}{L^2} \epsilon^2 \right), \tag{30}$$

and $-\frac{1}{24} \frac{\hbar \pi c}{L}$ is the Casimir energy of the initial vacuum state.

Then, until order ϵ^2, the total renormalised energy $\langle \hat{\mathbb{E}}_{\text{tot}}(t) \rangle \equiv \langle \hat{\mathbb{E}}_{\text{tot,dyn}}(t) \rangle + \langle \hat{\mathbb{E}}_{\text{tot,vac}}(t) \rangle$, can be splinted in the following way

$$\langle \hat{\mathbb{E}}_{\text{tot}}(t) \rangle = -\frac{1}{24} \frac{\hbar \pi c}{L} - \epsilon \int_0^t F_{\text{tot}}(\tau) \dot{g}(\tau)\, d\tau. \tag{31}$$

The first part is the static Casimir energy of the initial vacuum state, and the second part is minus the work done, until time t, by the reaction force

$$F_{\text{tot}}(t) \equiv \frac{1}{24}\frac{\hbar\pi c}{L^2} + \frac{\hbar\epsilon}{6\pi c^2}\sum_{k=0}^{N^*}\dddot{g}\left(t - \frac{2Lk}{c}\right) + \frac{\epsilon\hbar\pi}{6L^2}\left(\frac{1}{2}\dot{g}(t) + \sum_{k=1}^{N^*}\dot{g}\left(\tau - \frac{2Lk}{c}\right)\right). \tag{32}$$

Finally note that, in our situation (only one moving boundary), the term of order ϵ in (32) coincides with the formula (2) of the Jaekel and Reynaud (1993).

4. 3 + 1-DIMENSIONAL CASE

4.1. A Single Moving Mirror

We consider the massless Klein-Gordon field into the domains

$$\Omega_t^+ = [\epsilon g(t), L_1] \times [-L_2, L_2] \times [-L_3, L_3];$$
$$\Omega_t^- = [-L_1, \epsilon g(t)] \times [-L_2, L_2] \times [-L_3, L_3],$$

with $L_1, L_2, L_3 \gg 1$, and we take the following set of eigenfunctions

$$f_{\pm,\mathbf{n}}(t, \mathbf{x}; \epsilon) \equiv \sqrt{\frac{2}{L_1 \mp \epsilon g(t)}}\sin\left(n_1\pi\frac{x_1 - \epsilon g(t)}{L_1 \mp \epsilon g(t)}\right) \times \frac{1}{\sqrt{2L_2}}e^{i\frac{\pi n_2 x_2}{L_2}}\frac{1}{\sqrt{2L_3}}e^{i\frac{\pi n_3 x_3}{L_3}}, \tag{33}$$

with $n_1 \in \mathbb{N}$ and $n_2, n_3 \in \mathbb{Z}$. Then, until order ϵ^2 the dynamical energy is

$$\left\langle \hat{E}^0_{\pm,\text{dyn}}(t)\right\rangle = \frac{\epsilon^2\hbar L_2 L_3}{2(\pi c)^4}\int_{-\infty}^{\infty}d\omega_2\int_{-\infty}^{\infty}d\omega_3\int_0^{\infty} \times \int_0^{\infty}\frac{\omega_1^2(\omega_1')^2 d\omega_1 d\omega_1'}{\omega\omega'(\omega+\omega')}\left|\int_0^t \dot{g}(\tau)e^{i(\omega+\omega')\tau}d\tau\right|^2, \tag{34}$$

where

$$\omega_{\pm,\mathbf{n}}(t;\epsilon) = \sqrt{\left(\frac{\pi c n_1}{L_1 \mp \epsilon g(t)}\right)^2 + \left(\frac{\pi c n_2}{L_2}\right)^2 + \left(\frac{\pi c n_3}{L_3}\right)^2}$$

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}, \quad \omega' = \sqrt{(\omega_1')^2 + \omega_2^2 + \omega_3^2}.$$

Making the change of variables

$$\begin{cases} \omega_1 = u \sin\alpha' \cos\alpha (\sin\alpha + \sin\alpha')^{-1} \\ \omega_1' = u \sin\alpha \cos\alpha' (\sin\alpha + \sin\alpha')^{-1} \\ \omega_2 = u \sin\alpha \sin\alpha' (\sin\alpha + \sin\alpha')^{-1} \cos\beta \\ \omega_3 = u \sin\alpha \sin\alpha' (\sin\alpha + \sin\alpha')^{-1} \sin\beta, \end{cases} \tag{35}$$

we get

$$\left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t) \right\rangle = \frac{\epsilon^2 \hbar L_2 L_3}{2(\pi c)^4} \mathcal{R} \int_0^\infty du u^4 \left| \int_0^t \dot{g}(\tau) e^{iu\tau} d\tau \right|^2, \tag{36}$$

where a maple calculation provides

$$\mathcal{R} \equiv 2\pi \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^3\alpha \sin^3\alpha' \cos^2\alpha \cos^2\alpha'}{(\sin\alpha + \sin\alpha')^6} d\alpha \, d\alpha' = \frac{\pi}{90}.$$

If we assume that $g \in \mathcal{C}^4(\mathbb{R}) \cap \mathcal{C}^5[0, T]$, the frequency cut-off regularised dynamical energy is given by

$$\begin{aligned} \left\langle \hat{E}^0_{\pm,\mathrm{dyn}}(t;\gamma) \right\rangle = \frac{\epsilon^2 \hbar L_2 L_3}{90\pi^3 c^4} \Bigg[\frac{\dot{g}^2(t)}{\gamma^3} - \frac{1}{\gamma} \left(\dddot{g}(t)\dot{g}(t) - \frac{1}{2} \ddot{g}^2(t) \right) \\ + \frac{\pi}{2} \left(\ddddot{g}(t)\dot{g}(t) - \dddot{g}(t)\ddot{g}(t) + \int_0^t \dddot{g}^2(\tau)\, d\tau \right) \Bigg]. \end{aligned} \tag{37}$$

To renormalise this quantity we follow the same method as Gütig and Eberlein (1998) (see for more details the explanation of the formula (5.17) of Gütig and Eberlein (1998)). We suppose that the external density force that produces the movement of the wall has the form

$$\mathcal{F}_{\mathrm{ext}}(t) = \rho_1 \epsilon \ddot{g}(t) + \rho_2 \epsilon \ddddot{g}(t), \tag{38}$$

and we propose that

$$\rho_1 \equiv \rho_{\mathrm{exp}} - \frac{\hbar}{90\pi^3 c^4 \gamma^3}; \quad \rho_2 \equiv \frac{\hbar}{180\pi^3 c^4 \gamma}, \tag{39}$$

where ρ_{exp} is the experimental mass density of the moving wall. Thus, since the work done by the external density force is $4\epsilon L_1 L_2 \int_0^t d\tau \mathcal{F}_{\mathrm{ext}}(\tau)\dot{g}(\tau)$, the divergent terms of (37) are cancelled, and we can define the renormalised dynamical energy per unit of area by

$$\mathfrak{E}^0_{\pm,\mathrm{dyn}}(t) \equiv \frac{\epsilon^2 \hbar}{720\pi^2 c^4} \left(\ddddot{g}(t)\dot{g}(t) - \dddot{g}(t)\ddot{g}(t) + \int_0^t \dddot{g}^2(\tau)\, d\tau \right). \tag{40}$$

Remark 4.1. When $t \geq T$, the radiated energy per unit of area is

$$\mathfrak{E}^0_{\pm,\mathrm{dyn}}(t \geq T) = \frac{\epsilon^2 \hbar}{720\pi^2 c^4} \int_0^T \dddot{g}^{\,2}(\tau)d\tau, \tag{41}$$

in agreement with Ford and Vilenkin (1982).

In this case the regularised energy of the vacuum at time t is

$$\begin{aligned}\left\langle \hat{E}^0_{\pm,vac}(t;\gamma)\right\rangle &\equiv \lim_{L_1,L_2,L_3\to\infty} \sum_{\substack{n_1 \in \mathbb{N} \\ n_2, n_3 \in \mathbb{Z}}} \frac{\hbar}{2}\omega_{\pm,\mathbf{n}}(t;\epsilon)e^{-\gamma\omega_{\pm,\mathbf{n}}(t;\epsilon)} \\ &= \lim_{L_1,L_2,L_3\to\infty} \frac{6\hbar}{\pi^2 c^3 \gamma^4}(L_1 \mp \epsilon g(t))L_2 L_3, \end{aligned} \tag{42}$$

and we can conclude that the energy of the vacuum per unit volume, whose value is $\frac{3\hbar}{2\pi^2 c^3 \gamma^4}$, coincides with the energy density of the Minkowskian vacuum.

To finish this Section we review the results obtained by Ford and Vilenkin (1982). Using Green's functions the authors construct solutions that satisfy exactly the wave equation and, until order ϵ, the Dirichlet boundary condition.

With these solutions, applying the "point-splitting" ansatz, the authors calculate the regularised reaction force on the mirror per unit area (Eq. (4.12) of the Ford and Vilenkin (1982))

$$F(t;a) = \frac{\epsilon\hbar}{24\pi^2}\left(\frac{1}{a^3 c}\ddot{g}(t) - \frac{1}{10ac^3}\ddddot{g}(t) - \frac{1}{15c^4}g^{(5)}(t)\right), \tag{43}$$

where $0 < a \ll 1$ is a cut-off with dimensions of length.

Then, the total regularised dynamical energy per unit area is minus the work done by $F(t;a)$, that is,

$$\begin{aligned}\mathfrak{E}^0_{\text{tot, dyn}}(t;a)_{FV} &\equiv -\epsilon \int_0^t F(\tau;a)\dot{g}(\tau)\,d\tau \\ &= -\frac{\epsilon^2\hbar}{48\pi^2 a^3 c}\dot{g}^2(t) + \frac{\epsilon^2\hbar}{240\pi^2 ac^3}\left(\dddot{g}(t)\dot{g}(t) - \frac{1}{2}\ddot{g}^{\,2}(t)\right) \\ &\quad + \frac{\epsilon^2\hbar}{360\pi^2 c^4}\left(\ddddot{g}(t)\dot{g}(t) - \dddot{g}(t)\ddot{g}(t) + \int_0^t \dddot{g}^{\,2}(\tau)\,d\tau\right). \end{aligned} \tag{44}$$

Therefore, the total renormalised dynamical energy per unit area is defined by

$$\mathfrak{E}^0_{\text{tot, dyn}}(t)_{FV} \equiv \frac{\epsilon^2\hbar}{360\pi^2 c^4}\left(\ddddot{g}(t)\dot{g}(t) - \dddot{g}(t)\ddot{g}(t) + \int_0^t \dddot{g}^{\,2}(\tau)\,d\tau\right), \tag{45}$$

that coincides with $\mathfrak{E}^0_{+,\mathrm{dyn}}(t) + \mathfrak{E}^0_{-,\mathrm{dyn}}(t)$. Note that, the difference between (44) and $\frac{1}{4L_2L_3}(\langle \hat{E}^0_{+,\mathrm{dyn}}(t;\gamma)\rangle + \langle \hat{E}^0_{-,\mathrm{dyn}}(t;\gamma)\rangle)$ is due to the two different approaches used to obtain these quantities.

4.2. Two Mirrors

We study the massless Klein-Gordon field inside and outside of the domain $\Omega_t = [\epsilon g(t), L_1] \times [-L_2, L_2] \times [-L_3, L_3]$ with $L_2, L_3 \gg 1$. The regularised dynamical energy inside is

$$\left\langle \hat{E}^0_{\text{in, dyn}}(t;\gamma)\right\rangle \equiv \frac{\epsilon^2\hbar L_2L_3\pi^3c^2}{L_1^6}\sum_{n,k=1}^{\infty}\int_0^{\infty}\frac{du u\left|kn\int_0^t \dot{g}(\tau)e^{i(\omega_n(u)+\omega_k(u))(\tau+i\frac{\gamma}{2})}d\tau\right|^2}{\omega_n(u)\omega_k(u)(\omega_n(u)+\omega_k(u))}, \quad (46)$$

with $\omega_n(u) = \sqrt{\frac{\pi^2c^2n^2}{L_1^2} + u^2}$. And the regularised dynamical energy outside is given by the formula (37).

From the Abel-Plana formula (see the formula (2.31) of the Mostepanenko, and Trunov (1997)), we can show that the divergent part of $\langle \hat{E}^0_{\text{in, dyn}}(t;\gamma)\rangle$, namely $\langle \hat{E}^0_{D,\text{in, dyn}}(t;\gamma)\rangle$, is given by

$$\left\langle \hat{E}^0_{D,\text{in, dyn}}(t;\gamma)\right\rangle = \frac{\epsilon^2\hbar L_2L_3}{90\pi^3c^4}\left[\frac{\dot{g}^2(t)}{\gamma^3} - \frac{1}{\gamma}\left(\dddot{g}(t)\dot{g}(t) - \frac{1}{2}\ddot{g}^2(t)\right)\right], \quad (47)$$

that is, the total divergent dynamical energy of the system coincides with the total divergent dynamical energy when only it exist a single mirror. Consequently, this divergent quantity is removed assuming that the external density force is given by (38) and (39).

To calculate the energy of the vacuum at time t inside Ω_t, we make use of the following version to the Euler-Mclaurin formula

$$\sum_{n=1}^{\infty} F(n) = \int_0^x dx F(x) - \frac{1}{2}F(0) + 2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2\pi)^{2n+2}}F^{(2n+1)}(0)\zeta_R(2n+2), \quad (48)$$

that can be easily deduced from the Abel-Plana formula.

From (48), it is not difficult to prove that

$$\left\langle \hat{E}^0_{in,vac}(t;\gamma)\right\rangle = \frac{6(L_1-\epsilon g(t))L_2L_3\hbar}{\pi^2c^3\gamma^4} - \frac{L_2L_3\hbar}{\pi c^2\gamma^3} - \frac{L_2L_3\hbar\pi^2c}{360(L_1-\epsilon g(t))^3}. \quad (49)$$

Thus, we conclude that the total renormalised energy of the vacuum at time t per unit area, denoted by $\mathfrak{E}_{\text{vac}}(t)$, is

$$\mathfrak{E}_{\text{vac}}(t) \equiv -\frac{\hbar\pi^2 c}{1440(L_1 - \epsilon g(t))^3}. \tag{50}$$

5. THE BACK-REACTION PROBLEM

In this Section we consider the model that describes the interaction between a single moving mirror in the two dimensional space-time and the massless Klein-Gordon field.

Let $l_\epsilon(t)$ be the trajectory of the mirror. The Lagrangian of the field on the right and on the left side of the mirror, obtained from the Eqs. (4) and (7) of I, is given by (see Schützhold *et al.*, 1998)

$$L_{\pm,F}(t) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\dot{Q}^2_{\pm,n} - \omega^2_{\pm n}(t;\epsilon)Q^2_{\pm,n}\right) + \sum_{n,k=1}^{\infty} Q_{\pm,n}M_{\pm,nk}(t;\epsilon)\dot{Q}_{\pm,k}$$
$$+\frac{1}{2}\sum_{n,k,r=1}^{\infty} Q_{\pm,n}M_{\pm,nr}(t;\epsilon)M_{\pm,kr}(t;\epsilon)Q_{\pm,k}, \tag{51}$$

where in this case the frequencies are $\omega_{\pm,n}(t;\epsilon) = \frac{\pi cn}{L\mp l_\epsilon(t)}$ with $L \gg 1$, and

$$M_{\pm,nk}(t;\epsilon) = \begin{cases} \mp\frac{\dot{l}_\epsilon(t)}{L\mp l_\epsilon(t)}\frac{2nk}{k^2-n^2} & n \neq k \\ 0 & n = k. \end{cases}$$

The Lagrangian that describes the movement of the boundary is

$$L_B(t) = \frac{M}{2}\dot{l}^2_\epsilon - W(l_\epsilon, t), \tag{52}$$

where W is the prescribed potential energy function, and M is the mass of the boundary.

The Lagrangian $L_T \equiv L_{+,F} + L_{-,F} + L_B$ describes completely the interaction between the field and the mirror. Then the corresponding Euler-Lagrange equations give the coupled system

$$\ddot{Q}_{\pm,n} = -\omega^2_{\pm,n}(t;\epsilon)Q_{\pm,n} + 2\sum_{k=1}^{\infty} M_{\pm,nk}(t;\epsilon)\dot{Q}_{\pm,k}$$
$$+\sum_{k=1}^{\infty}\dot{M}_{\pm,nk}(t;\epsilon)Q_{\pm,k} + \sum_{k,r=1}^{\infty} M_{\pm,nr}(t;\epsilon)M_{\pm,kr}(t;\epsilon)Q_{\pm,k}. \tag{53}$$

$$M\ddot{l}_\epsilon = -\partial_x W(l_\epsilon, t) + F_{\text{mech}}, \tag{54}$$

where F_{mech} is the classical reaction force, and its value is (see Appendix B)

$$\begin{aligned} F_{\text{mech}} &= \lim_{\delta\to 0}\left[\mathcal{E}^0(t, l_\epsilon(t) - |\delta|) - \mathcal{E}^0(t, l_\epsilon(t) + |\delta|)\right] \\ &= -\frac{1}{L - l_\epsilon}\sum_{n,k=1}^{\infty} \omega_{+,n}(t;\epsilon)\omega_{+,k}(t;\epsilon) Q_{+,n} Q_{+,k} \\ &\quad + \frac{1}{L + l_\epsilon}\sum_{n,k=1}^{\infty} \omega_{-,n}(t;\epsilon)\omega_{-,k}(t;\epsilon) Q_{-,n} Q_{-,k}, \end{aligned} \tag{55}$$

where $\mathcal{E}^0$ is the energy density of the massless Klein-Gordon field (Eq. (6) of I).

The conjugated momenta are

$$P_{\pm,n} \equiv \frac{\partial L_T}{\partial \dot{Q}_{\pm,n}} = \dot{Q}_{\pm,n} - \sum_{k=1}^{\infty} M_{\pm,nk}(t;\epsilon) Q_{\pm,k} \tag{56}$$

$$p_\epsilon \equiv \frac{\partial L_T}{\partial \dot{l}_\epsilon} = M\dot{l}_\epsilon - \frac{1}{\dot{l}_\epsilon}\sum_{n,k=1}^{\infty}\left(M_{+,nk}(t;\epsilon)P_{+,n}Q_{+,k} + M_{-,nk}(t;\epsilon)P_{-,n}Q_{-,k}\right), \tag{57}$$

and the full Hamiltonian is

$$H_T(t) \equiv \frac{M}{2}\dot{l}_\epsilon^2 + W(l_\epsilon, t) + E_{+,F} + E_{-,F}, \tag{58}$$

where $E_{+,F}$ (resp. $E_{-,F}$) is the energy of the field on the right (resp. on the left) of the mirror.

The expression (58) can be written as follows

$$\begin{aligned} H_T(t) &= \frac{1}{2M}\left(p_\epsilon + \sum_{n,k=1}^{\infty} g_{nk}\left(\frac{P_{-,n}Q_{-,k}}{L + l_\epsilon} - \frac{P_{+,n}Q_{+,k}}{L - l_\epsilon}\right)\right)^2 + W(l_\epsilon, t) \\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty}\left(P_{+,n}^2 + \omega_{+,n}^2(t;\epsilon)Q_{+,n}^2\right) + \frac{1}{2}\sum_{n=1}^{\infty}\left(P_{-,n}^2 + \omega_{-,n}^2(t;\epsilon)Q_{-,n}^2\right), \end{aligned} \tag{59}$$

where we have introduced

$$g_{nk} = \begin{cases} \frac{2nk}{k^2 - n^2} & n \neq k \\ 0 & n = k. \end{cases}$$

The quantum Hamiltonian, denoted by $\hat{H}_T(t)$, is obtained following the well-known canonical quantisation procedure. Then the Schrödiger equation

$$\begin{cases} i\hbar\partial_t|\Phi\rangle_t = \hat{H}_T(t)|\Phi\rangle_t \\ |\Phi\rangle_0 = |0_T\rangle, \end{cases} \tag{60}$$

describes the interaction between the field and the mirror, where $|0_T\rangle$ is the vacuum state of the full system.

Remark 5.1. In Law (1995) the author obtains the quantum equation, that describes the back-reaction, inside a bounded $1+1$-dimensional cavity. It is clear, from our analysis, that the field outside the cavity must be included in the full quantum equation.

We believe that the study of the quantum Eq. (60) is very complicated. For this reason we consider the following semi-classical model that describes the back-reaction:

We only quantise the dynamical variables of the field, and we assume that the dynamical variable l_ϵ satisfies the equation

$$M\ddot{l}_\epsilon = -\partial_x W(l_\epsilon, t) + \langle F_{\text{mech}}(t;\gamma)\rangle, \tag{61}$$

where

$$\langle F_{\text{mech}}(t;\gamma)\rangle = \langle \hat{\mathcal{E}}^0_-(t, l_\epsilon;\gamma)\rangle - \langle \hat{\mathcal{E}}^0_+(t, l_\epsilon;\gamma)\rangle \tag{62}$$

is the frequency cut-off regularised reaction force produced by the evolved vacuum state (see Appendix B).

Remark 5.2. It is important to emphasise the fact that the Eq. (62) must be understood as the second Newton's law, and then we can replace the force $-\partial_x W(l_\epsilon, t)$ by a more general classical external force (dissipative force, friction force, ...). Obviously, we cannot make this substitution in Eq. (60), because this equation comes from a Hamiltonian system.

The linear term on l_ϵ of the cut-off frequency regularised reaction force is obtained inserting (12) into (62), then the Eq. (61) becomes

$$M\ddot{l}_\epsilon = -\partial_x W(l_\epsilon, t) + \frac{\hbar}{6c^2\pi}\left(\dddot{l}_\epsilon(t) - \frac{2}{\pi\gamma}\ddot{l}_\epsilon(t)\right). \tag{63}$$

After the mass renormalisation $M = M_{\text{exp}} - \frac{\hbar}{3c^2\pi^2\gamma}$ (Eq. (8)), we obtain the equation

$$M_{\text{exp}}\ddot{l}_\epsilon = -\partial_x W(l_\epsilon, t) + \frac{\hbar}{6c^2\pi}\dddot{l}_\epsilon(t). \tag{64}$$

Note that the reaction force that appears in (64) must be understood as a small perturbation, because this force is obtained assuming that l_ϵ is a prescribed trajectory that satisfies the equation $M_{\exp}\ddot{l}_\epsilon = -\partial_x W(l_\epsilon, t)$, and clearly do not play the same role as the external forces. Note also that with this interpretation the runaway solutions are eliminated.

Effectively, if we write $l_\epsilon = \epsilon g + \tilde{x}$, where ϵg is the solution of the Newton's equation $\epsilon\,\ddot{g} = -\partial_x W(\epsilon g, t)$, and $\tilde{x}$ is the induced movement of the boundary due to the evolved vacuum state. Since, $|\tilde{x}|$ must be smaller than $\epsilon|g|$, making the approximation $\partial_x W(l_\epsilon, t) \approx \partial_x W(\epsilon g, t)$ the Eq. (64) behaves

$$M_{\exp}\ddot{\tilde{x}} \approx \frac{\hbar\epsilon}{6c^2\pi}\,\dddot{g}\,(t), \tag{65}$$

and we deduce that the trajectory of the mirror is

$$l_\epsilon(t) \approx \epsilon g(t) + \frac{\hbar\epsilon}{6M_{\exp}c^2\pi}\dot{g}(t). \tag{66}$$

Finally inserting this expression in (10), we conclude that the total radiated energy, until order ϵ^2, in this approximation is

$$\frac{\hbar\epsilon^2}{6c^2\pi}\int_0^T \left(\ddot{g}^{\,2}(\tau) + \frac{\hbar^2}{36M_{\exp}^2c^4\pi^2}\,\dddot{g}^{\,2}(\tau)\right)d\tau. \tag{67}$$

6. CONCLUSIONS

In these two papers we have showed that the dynamical Casimir effect in cavities with perfect reflecting boundaries presents several difficulties: The concept of particle is ill-defined when the boundary moves, a divergent production of particles is possible when the movement of the boundary has some type of discontinuities, and the renormalised dynamical energy is not positive when the boundary moves.

We have also showed that, from the Hamiltonian approach, the regularised Casimir energy can be calculated in a very easy way. In particular, in the $1+1$-dimensional case, we have calculated explicitly the energy when there is two boundaries.

Finally in the last Section we have calculated the radiation energy emitted by a mirror when their recoil is taken into account.

APPENDIX A

Here we consider the massive Klein-Gordon in two dimensions, and we assume that the field vanish in a prescribed trajectory $(t, \epsilon g(t))$.

Until order ϵ^2 we have

$$\left\langle \hat{E}^m_{\pm,\mathrm{dyn}}(t)\right\rangle = \frac{\epsilon^2}{2\pi^2 c^2 \hbar^5}\int_0^\infty\int_0^\infty \frac{x^2y^2\left|\int_0^t \dot{g}(\tau)e^{i(\omega(x)+\omega(y))\tau}d\tau\right|^2}{\omega(x)\omega(y)(\omega(x)+\omega(y))}\,dx\,dy, \quad (68)$$

where $\omega(x) = \frac{1}{\hbar}\sqrt{x^2+m^2c^4}$.

The part of the energy that contains the divergent terms is

$$\left\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t)\right\rangle \equiv \frac{\epsilon^2\dot{g}^2(t)}{2\pi^2 c^2 \hbar^5}\int_0^\infty\int_0^\infty \frac{x^2y^2}{\omega(x)\omega(y)(\omega(x)+\omega(y))^3}\,dx\,dy. \quad (69)$$

Now we define

$$\left\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\right\rangle \equiv \frac{\epsilon^2\dot{g}^2(t)}{2\pi^2 c^2 \hbar^5}\int_0^\infty\int_0^\infty \frac{x^2y^2e^{-(\omega(x)+\omega(y))\gamma}}{\omega(x)\omega(y)(\omega(x)+\omega(y))^3}\,dx\,dy. \quad (70)$$

Making the change of variables

$$u = \frac{1}{m^2c^4}\sqrt{x^2+m^2c^4}; \quad v = \frac{1}{m^2c^4}\sqrt{y^2+m^2c^4},$$

we obtain

$$\begin{aligned}\left\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\right\rangle &= \frac{\epsilon^2\dot{g}^2(t)m}{2\pi^2}\int_1^\infty\int_1^\infty \frac{\sqrt{u^2-1}\sqrt{v^2-1}e^{-\frac{mc^2}{\hbar}\gamma(u+v)}}{(u+v)^3}\,du\,dv \\ &= \frac{\epsilon^2\dot{g}^2(t)m}{2\pi^2}\int_1^\infty\int_1^\infty \frac{uve^{-\frac{mc^2}{\hbar}\gamma(u+v)}}{(u+v)^3}\,du\,dv \\ &\quad + \frac{\epsilon^2\dot{g}^2(t)m}{2\pi^2}\int_1^\infty\int_1^\infty \frac{(\sqrt{u^2-1}\sqrt{v^2-1}-uv)e^{-\frac{mc^2}{\hbar}\gamma(u+v)}}{(u+v)^3} \\ &\quad \times du\,dv \equiv (A)+(B). \end{aligned} \quad (71)$$

Since

$$\left|\frac{(\sqrt{u^2-1}\sqrt{v^2-1}-uv)}{(u+v)^3}\right| \le \frac{2}{u^{\frac{3}{2}}v^{\frac{3}{2}}},$$

from the Lebesgue's Dominated Convergence Theorem we deduce that

$$\lim_{\gamma\to 0}(B) = \frac{\epsilon^2\dot{g}^2(t)m}{2\pi^2}\int_1^\infty\int_1^\infty \frac{(\sqrt{u^2-1}\sqrt{v^2-1}-uv)}{(u+v)^3}\,du\,dv = \mathcal{O}(mc^2). \quad (72)$$

Now we study the integral $\int_1^\infty \int_1^\infty \frac{uve^{-\frac{mc^2}{\hbar}\gamma(u+v)}}{(u+v)^3}\,du\,dv$. Making the change $\bar{u} = \gamma u$ and $\bar{v} = \gamma v$ we can write

$$
\begin{aligned}
\int_1^\infty \int_1^\infty \frac{uve^{-\frac{mc^2}{\hbar}\gamma(u+v)}}{(u+v)^3}\,du\,dv &= \frac{1}{\gamma}\int_0^\infty \int_0^\infty \frac{\bar{u}\bar{v}e^{-\frac{mc^2}{\hbar}(\bar{u}+\bar{v})}}{(\bar{u}+\bar{v})^3}\,d\bar{u}\,d\bar{v} \\
&\quad - \frac{1}{\gamma}\int_0^\gamma \int_0^\gamma \frac{\bar{u}\bar{v}e^{-\frac{mc^2}{\hbar}(\bar{u}+\bar{v})}}{(\bar{u}+\bar{v})^3}\,d\bar{u}\,d\bar{v} \\
&\quad - \frac{2}{\gamma}\int_0^\gamma d\bar{v}\int_\gamma^\infty d\bar{u}\,\frac{\bar{u}\bar{v}e^{-\frac{mc^2}{\hbar}(\bar{u}+\bar{v})}}{(\bar{u}+\bar{v})^3} \\
&= \frac{1}{6\gamma}\frac{\hbar}{mc^2} - \frac{5}{4} + \mathcal{O}\left(\frac{mc^2}{\hbar}\gamma\right). \qquad (73)
\end{aligned}
$$

Consequently, we have

$$
\left\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\right\rangle = \frac{\epsilon^2\dot{g}^2(t)\hbar}{12\pi^2c^2\gamma} + \mathcal{O}\left(\frac{mc^2}{\hbar}\gamma\right) + \mathcal{O}(mc^2). \qquad (74)
$$

Now we write $\langle \hat{E}^m_{\pm,\mathrm{dyn}}(t;\gamma)\rangle = \langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\rangle + \langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t;\gamma)\rangle$. An easy calculation shows that

$$
\lim_{m\to\infty}\lim_{\gamma\to 0}\left\langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t;\gamma)\right\rangle = 0.
$$

However, the convergent part of $\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\rangle$, that is, $\mathcal{O}(\frac{mc^2}{\hbar}\gamma) + \mathcal{O}(mc^2)$, diverges when $\gamma \to 0$ and $m \to \infty$. For this reason, if we impose that the renormalised part of $\langle \hat{E}^m_{\pm,\mathrm{dyn}}(t)\rangle$, namely $\langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t)\rangle$, satisfies the condition $\lim_{m\to\infty}\langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t)\rangle = 0$, we must define

$$
\left\langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t)\right\rangle \equiv \lim_{\gamma\to 0}\left(\left\langle \hat{E}^m_{\pm,\mathrm{dyn}}(t;\gamma)\right\rangle - \left\langle \hat{E}^m_{D,\pm,\mathrm{dyn}}(t;\gamma)\right\rangle\right). \qquad (75)
$$

Now, it is not difficult to prove that

$$
\lim_{m\to 0}\left\langle \hat{\mathbb{E}}^m_{\pm,\mathrm{dyn}}(t)\right\rangle = \left\langle \hat{\mathbb{E}}^0_{\pm,\mathrm{dyn}}(t)\right\rangle. \qquad (76)
$$

This shows the consistency of the frequency cut-off regularisation, in the massless case.

APPENDIX B

In this Appendix we calculate the force that acts on a moving boundary due to the massless Klein-Gordon field (the reaction force).

In $1 + 1$-dimensions the stress tensor of the field is

$$\begin{pmatrix} \mathcal{E} & \frac{1}{c}\mathcal{F} \\ \frac{1}{c}\mathcal{F} & \mathcal{E} \end{pmatrix}, \tag{77}$$

where $\mathcal{E}$ is the energy density of the system (Eq. (6) of I), and $\mathcal{F} \equiv -\hbar^2 c^2 \partial_t \phi \partial_x \phi$ is the energy flux.

The momentum of the field is $P \equiv \frac{1}{c^2} \int_{\mathbb{R}} dx \mathcal{F}$. Then, if ΔP_{mech} is the momentum increment of the moving boundary due to the interaction with the Klein-Gordon field, the momentum conservation law provides

$$\partial_t \left(P + \Delta P_{\text{mech}} \right) = 0,$$

consequently the classical reaction force $F_{\text{mech}} \equiv \partial_t (\Delta P_{\text{mech}})$ is given by

$$F_{\text{mech}} = -\frac{1}{c^2} \partial_t \int_{\mathbb{R}} dx \mathcal{F} = \hbar^2 \partial_t \int_{\mathbb{R}} dx \partial_t \phi \partial_x \phi = \partial_t \int_{\mathbb{R}} dx \xi \partial_x \phi, \tag{78}$$

where we have used the canonically conjugated momentum of the field (Eq. (5) of I).

In the particular case of a single moving mirror with trajectory $(t, q(t))$ we easily find

$$F_{\text{mech}}(t) = \lim_{\delta \to 0} \left[\mathcal{E}^0(t, q(t) - |\delta|) - \mathcal{E}^0(t, q(t) + |\delta|) \right], \tag{79}$$

where $\mathcal{E}^0$ is the energy density of the massless Klein-Gordon field (see Eq. (6) of I).

Finally note that the reaction force produced by the evolved vacuum state (the so-called radiation reaction force) is

$$\langle F_{\text{mech}} \rangle \equiv \partial_t \int_{\mathbb{R}} dx \langle 0 | (\mathcal{T}^t)^\dagger \hat{\xi} \partial_x \hat{\phi} \mathcal{T}^t | 0 \rangle. \tag{80}$$

In the case of a single moving mirror, the frequency cut-off regularised reaction force is given by

$$\langle F_{\text{mech}}(t; \gamma) \rangle = \langle \hat{\mathcal{E}}^0_-(t, l_\epsilon; \gamma) \rangle - \langle \hat{\mathcal{E}}^0_+(t, l_\epsilon; \gamma) \rangle, w \tag{81}$$

where $\langle \hat{\mathcal{E}}^0_\pm(t, l_\epsilon; \gamma) \rangle$ is the regularised energy density in both sides of the mirror (see Eq. (12)).

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